

# RESEARCH STATEMENT

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## 1. INTRODUCTION

My research interests lie in Combinatorics, and; in particular to utilize the tools of algebra and category theory to illuminate the fundamental structure of combinatorial objects. Much of the mathematical advances of the past century have been in finding underlying connections and functorial relationships between the branches of algebra, geometry and topology. By applying this strategy to Combinatorics, we hope to advance the study if this branch of mathematics. As relatively young branch, this approach should be fruitful as we push in this direction. My work currently revolves around developing a  $\times$ -homotopy theory for the category of graphs.

As a Teacher-Scholar, I am also interested and engaged in expository works, and the scholarship of Teaching & Learning, as well as Outreach work.

## 2. $\times$ -HOMOTOPY IN GRAPHS

Homotopy Theory is an integral part of the study of Topology, and is a critical tool in describing topological properties and structure. It is natural then to generalize the idea of homotopy into other settings. One could then ask: “How might Homotopy Theory generalize to Graph Theory?” This is an interesting question since homotopies deal with continuous deformation, and graphs fundamentally discrete objects. One way homotopy theory has been applied to graphs is to treat graphs as simplicial objects, but this is merely an extension of the classical homotopy theory. Other attempts to develop a graph homotopy theory is to do so internally without explicit reference to topological constructions. The way this is done then is categorically. In this formulation, called  $\times$ -homotopy [7], we replace the notion of continuous function in topology, with **graph homomorphism**. Then, just as a homotopy is defined as a continuous function on a product of spaces, a  $\times$ -homotopy is defined as a graph homomorphism on a product of graphs.

More formally, we give a definition of  $\times$ -homotopy:

**Definition 2.1.** [7] Given  $f, g : G \rightarrow H$ , we say that  $f$  is  **$\times$ -homotopic** to  $g$ , written  $f \simeq g$ , if there is a map  $\Lambda : G \times I_n^\ell \rightarrow H$  such that  $\Lambda|_{G \times \{0\}} = f$  and  $\Lambda|_{G \times \{n\}} = g$ . We will say  $\Lambda$  is a length  $n$  homotopy.

What is interesting about this setting is that while the objects may seem similar to some of the objects of topology, the behavior of the category is quite different. For example the product of connected spaces is always a connected space, but the product of two connected graphs need not be a connected graph! This and other distinctions make it an interesting problem to see what topological concepts even have analogues in graphs, and if they do, how they might differ. This question is the driving force behind this project, a collaboration between myself, and a homotopy theorist, Dr. Laura Scull.

One of our first initial results was a simplification of the definition of homotopy. In classical topology, while a homotopy has a technical formal definition, we understand it to be a continuous deformation of a function into another function. In graphs, we showed that a  $\times$ -homotopy can be understood as a transformation of one graph homomorphism to another, by moving one vertex at a time. We call these transformations **spider moves** [5].

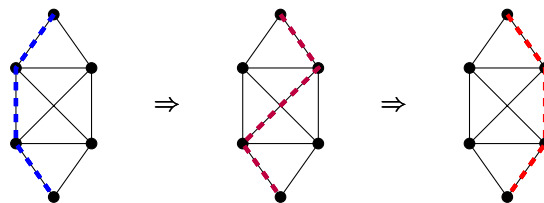


FIGURE 1. A homotopy between two paths by adjusting one vertex at a time.

Describing homotopies as spider moves greatly illuminates their behavior and simplifies the verification of these results.

Using this reformulation of homotopy, we've established that  $\mathbf{Gph}$  forms a **2-category** with  $\times$ -homotopy, and that its quotient forms the **homotopy category** [5]. We have also found an analogue for the **fundamental groupoid**, a central algebraic object in the study of classic homotopy, and showed that it shares many of the useful properties of the topological fundamental groupoid. Most importantly that homotopy equivalent graphs give equivalent groupoids [4]. Graph covers have been studied [1, 2, 8–10], but by treating graphs as simplicial spaces. Since one of the key properties of covers in topology is the lifting of homotopies, we found necessary and sufficient conditions for a graph to be a homotopy cover of another graph [6]. We then established a **Universal Cover** of a graph, a homotopy cover which homotopy covers every other homotopy cover of the graph. We then developed an analogue of deck transformations and covers like in topology [6]. Our next step is tackling higher homotopy groups for graphs. We've barely begun to scratch the surface of all the homotopy properties that could have graph analogues, so this will be an exciting and fruitful project for years to come.

**2.1. Undergraduate Research.** This area of math is relatively unexplored, but has an existing well-established analogue in topology, the underlying objects are finite and concrete, and the spider moves makes understanding of a homotopy's behavior very tractable. These properties makes our area particularly suited for undergraduate research. Understanding this, Dr. Scull and I applied for and received the Center for Undergraduate Research in Mathematics (CURM) mini-grant for the 20-21 school year. This funded a team of undergraduate scholars from each group to work on  $\times$ -homotopy problems. My students focused on homotopies of endomorphisms of graphs, in particular, of automorphisms. In their project, they showed that given a graph  $G$  the collection of automorphisms homotopic to the identity,  $\text{Null}(G)$ , formed a normal subgroup. Then, given the homotopy equivalences between a graph and its pleat:  $\rho : G \rightarrow \mathcal{P}\ell(G), \iota : \mathcal{P}\ell(G) \rightarrow G$ , that the map:  $\Phi : \text{Aut}(G) \rightarrow \text{Aut}(\mathcal{P}\ell(G)), \alpha \mapsto \rho\alpha\iota$  was a group homomorphism whose kernel was  $\text{Null}(G)$ . Future directions here could include exploring the automorphism groups of various pleats, and to identify concretely the relation between  $\text{Aut}(G), \text{Null}(G)$  and  $\text{Aut}(\mathcal{P}\ell(G))$ . They presented their work in an online math seminar (due to COVID) <https://www.youtube.com/watch?v=hylQFTKHDyQ> and produced a jupyter notebook to show concrete constructions of the objects they worked with: [Jupyter Notebook](#).

In our application for the CURM grant, we identified many potential avenues for student research. These included taking families of graphs and looking for patterns either in their homotopy covers, or their related homotopy constructions such as fundamental groups and group(oid)s or pleats. Given the concrete nature of these objects, writing code to generate examples is a good project for students so inclined and useful in looking for patterns.

### 3. OTHER PROJECTS

As someone very passionate about teaching and outreach, I am also interested in the scholarship of teaching & learning mathematics, as well as the scholarship of outreach. My first paper on teaching undergraduate mathematics was accepted by *PRIMUS* [3], detailing how one can facilitate an Inquiry course asynchronously online, or use asynchronous elements to supplement a synchronous of face-to-face course. I'm also currently involved in research projects to study the effects of Mastery/Stadards Based Grading on student Anxiety during pandemic, and an NSF funded project to study the efficacy of Team Based Inquiry Learning on lower division math courses. As I expand my teaching range to explore new pedagogical techniques, I am simultaneously interested in contributing to the literature on how one might implement these techniques effectively, or what impacts they may have on students.

Similarly, I wish to help establish the scholarship of math outreach. I was an invited author on the inaugural issue of the Journal of Math Circles, and have done a number of reviews for them. When later I was invited to join the editorial board, I was enthusiastic about it, since this was an opportunity to help grow this area of literature. People doing outreach work, just as teachers in the classroom, can benefit greatly from the work of scholars establishing the practices of effective outreach. Outreach work is something that's very important to me, and it's important to me to contribute to a body of work that will help other outreach facilitators develop and grow their own efforts.

I am a pure mathematician, but I enjoy collaborating with faculty from different disciplines, and exploring interesting applications of mathematics. I was asked to run non-parametric simulations in R by a biologist, looking at misregulation of genes exposed to high level of histones, compared to those that were not. The results were reported in a paper currently under review. I've also written a paper on using the PageRank algorithm as a mechanism to rank boxers or sports team, which has been accepted for publication. These projects have allowed me to step outside of pure math and see some of the interesting and valuable things that working with mathematics can provide. They also expand my repertoire as a teacher and mentor. PageRank and sports ranking have been discussed in my Linear Algebra course, and I've used what I've learned for the simulations in writing my own stats material. Having a broad base of knowledge is vital to student driven undergraduate work. Students have their own interests, and rather than conform to what is our own expectations, we need to meet them where they are, and show how math is present in this context. Continued applied and interdisciplinary collaboration will expand my ability to do this.

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